

From fundamental strings to small black holes

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ABSTRACT: We give evidence in favour of a string/black hole transition in the case of BPS fundamental string states of the Heterotic string. Our analysis goes beyond the counting of degrees of freedom and considers the evolution of dynamical quantities in the process. As the coupling increases, the string states decrease their size up to the string scale when a small black hole is formed. We compute the absorption cross section for several fields in both the black hole and the perturbative string phases. At zero frequency, these cross sections can be seen as order parameters for the transition. In particular, for the scalars fixed at the horizon the cross section evolves to zero when the black hole is formed.

KEYWORDS: Black Holes in String Theory, Black Holes.

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Contents

1.	Introduction	1
2.	Black hole phase	4
2.1	Review of the geometry	4
2.2	Absorption cross section	7
2.2.1	Equations of motion	7
2.2.2	Cross section	10
3.	String phase	13
3.1	String states	13
3.2	Absorption amplitude	15
3.2.1	Fermionic states	18
3.3	Absorption cross section	18
3.4	String size	19
3.4.1	Self-gravitating string	21
4.	Conclusion	22
A.	Barrier penetration at low energies	23
B.	Averages over string states	27
C.	Average string size	29

1. Introduction

Since its original proposal [1], the identification of fundamental strings with black holes has been widely recognized as a major challenge. The basic idea is to consider a Schwarzschild black hole of mass M and entropy $S \sim GM^2$. If we decrease the gravitational coupling while keeping the entropy fixed, the hole radius will decrease to the string scale. At this point, the mass, which increased in the process, is related to the gravitational coupling by $G_*M_* \sim \sqrt{\alpha'}$. Furthermore, at this point, the entropy $S \sim \sqrt{\alpha'}M_*$, computed for a free string of mass M_* , matches the black hole entropy. For smaller values of the coupling the black hole evolves into a weakly interacting string phase.

A major obstacle for a quantitative understanding of the string/black hole transition is the uncontrollable self-energy corrections to the string mass in the process of increasing the coupling, i.e. of forming the hole [2, 3]. BPS states have, on the other hand, a protected mass so that the next logical step would be to use them to make precise quantitative

statements about the transition. However, supergravity black hole solutions corresponding to BPS strings are quite peculiar. These geometries have a horizon radius of the order of the string length, so that α' corrections to the supergravity effective action can not be neglected [4]. In the particular case of four-dimensional black holes of the Heterotic string, the corrected near horizon geometry is $AdS_2 \times S^2$ [5] and the corresponding Wald entropy [6], which includes the contribution of higher curvature terms, reproduces the microscopic string degeneracy [7]. The main goal of this work is to go beyond this counting of degrees of freedom and to analyse the evolution of dynamical quantities in the string/black hole transition. We shall study the absorption cross section for various fields, as well as the evolution of the size of string states.

We consider a BPS state of the Heterotic string compactified on $S^1 \times T^5$ carrying Kałuza–Klein and winding quantum numbers (n, m) along the S^1 direction. In this controlled setup, let us briefly describe the string/black hole transition, as shown in figure 1. In the free string phase, for large values of the level $N = nm + 1$, we will show that the string average size is given by

$$\langle \mathcal{R}^2 \rangle \simeq \frac{\pi}{12} \alpha' \sqrt{N},$$

indicating that the usual random walk picture for very massive strings also holds for BPS states. Turning on the string coupling g , the \sqrt{N} bits of the string will start to interact gravitationally, reducing the string size. We shall see that, above $g^2 \sim N^{-3/4}$, the string is described by a self-gravitating string phase [3], with

$$\langle \mathcal{R}^2 \rangle \sim \frac{\alpha'}{g^4 N}.$$

Finally, above $g^2 \sim N^{-1/2}$, the large gravitational interaction collapses the string into a small black hole with fixed horizon radius (in the string frame) [7]

$$\mathcal{R}_h^2 \sim \alpha',$$

which is independent of the charges (n, m) and of the string coupling.

Further evidence for the string/black hole transition comes from the computation of the absorption cross section. In the black hole phase we compute the low frequency behaviour of the absorption cross section for a minimally coupled scalar and for a fixed scalar¹ obtaining, respectively,

$$\sigma_{abs} \simeq c G_N S, \quad \sigma_{abs} \sim \omega^\beta,$$

where c and $\beta > 0$ are numerical constants of order 1, G_N is the four-dimensional Newton constant and S is the hole entropy. These results are valid to *all* orders in α' . On the other hand, to leading order in perturbation theory we determine the absorption cross section for *all* massless particles of the Heterotic string, finding the universal result

$$\sigma_{abs} \simeq \frac{\pi}{2} g^2 \alpha' \sqrt{N} \simeq G_N S.$$

¹The fixed scalar result is restricted to a particular background with constant S^1 radius.

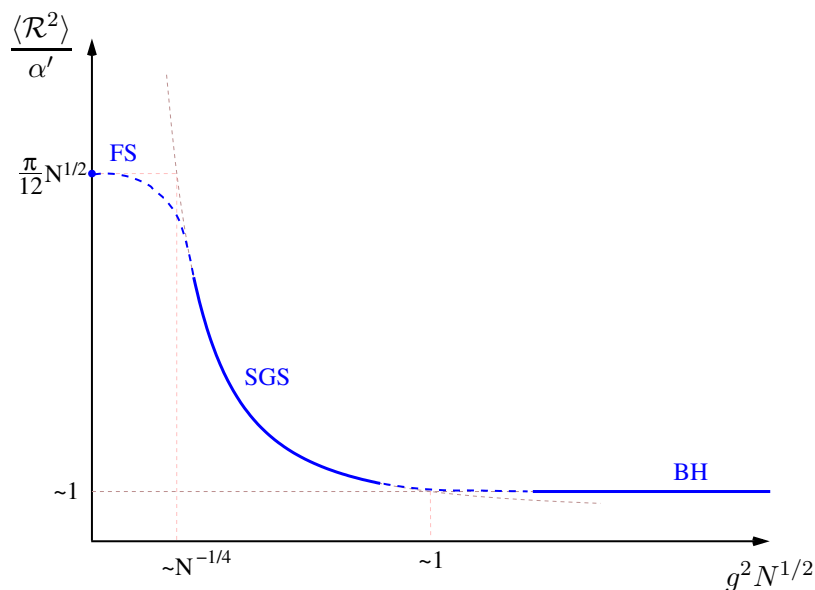


Figure 1: Evolution of the string average size through the transition. The free string (FS) regime is appropriate for extremely small coupling $g^2 \ll N^{-3/4}$. In the other extreme, for $g^2 \gg N^{-1/2}$, the black hole (BH) phase prevails. In between there is a self-gravitating string (SGS) phase.

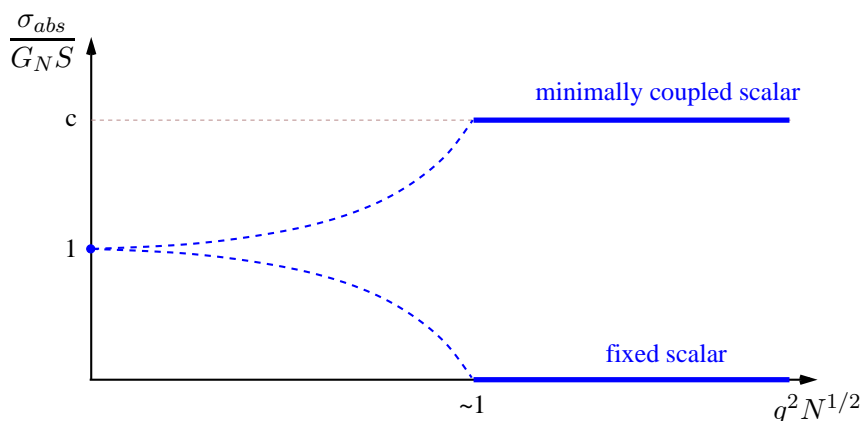


Figure 2: Absorption cross section at $\omega = 0$ as an order parameter. Once a black hole is formed the fixed scalar absorption cross section vanishes and the minimally coupled scalar absorption cross section is proportional to the hole entropy times the Newton constant. On the other hand, the perturbative string computation yields the same result for *all* massless particles. The precise behaviour of the cross section for all values of the coupling is not known (as indicated by the dashed lines).

These results support the physical picture explained above. Moreover, the vanishing of the absorption cross section at $\omega = 0$ for the fixed scalar is directly related to the existence of a black hole and to the attractor mechanism. Indeed, these scalars have their values at the horizon fixed by the hole charges so that their oscillations, and therefore the infalling flux, are suppressed, leading to the absence of absorption. Since this is a generic behaviour provided there is a horizon, we propose $\sigma_{abs}(\omega = 0)$ as an order parameter for the string/black hole (phase) transition (see figure 2).

Finally, we shall see that both self-gravitating string and black hole phases can be

studied in the classical limit $g \rightarrow 0$ and $N \rightarrow \infty$ with $g^2\sqrt{N}$ fixed. Therefore, as explained in the Conclusion, one should be able to see the transition within the gravity framework.

In section 2 we review the Heterotic BPS black hole geometry and compute the absorption cross section for both fixed and minimally coupled scalars in the gravity description. Section 3 is devoted to the perturbative string CFT description. Firstly we compute the absorption cross section for all massless fields. Then, at zero string coupling, we calculate the average size of the string states corresponding to the black hole. We also estimate the string size for small coupling. In section 4 we give our conclusions. Finally, in appendix A we study the low energy tunneling through a general one-dimensional Schrödinger potential, which is the problem one effectively needs to solve to study generic low energy scattering processes in spherically symmetric backgrounds. In appendices B and C some longer calculations concerning the ensemble of string microstates are performed.

For earlier work on the relation between strings and BPS black holes see [8–11]. Recent work on the subject includes [12–29].

2. Black hole phase

2.1 Review of the geometry

We consider a black hole in Heterotic string theory compactified on $S^1 \times T^5$. The only non-vanishing ten-dimensional fields are the string metric G_{ab} , the dilaton field Φ and the NS 2-form gauge potential B_{ab} . Neglecting α' corrections, the ten-dimensional effective action for these fields is

$$\frac{1}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[R + 4(\partial\Phi)^2 - \frac{1}{12} (dB)^2 \right],$$

where the Newton constant is related to the string coupling and tension by

$$16\pi G_N^{(10)} = (2\pi)^7 g_s^2 \alpha'^4.$$

In our conventions the dilaton field vanishes at infinity, so that the Newton constant has an explicit g_s^2 factor included.

The black hole will describe a fundamental string wrapping the S^1 circle, with Kaluza-Klein momentum and winding numbers (n, m) . Therefore, we consider a compactification to four dimensions with the ansatz

$$ds_{10}^2 = ds^2 + e^{2\lambda} (dx^4 + A_\mu dx^\mu)^2 + e^{2\nu} ds^2 (\mathbb{T}^5),$$

$$2\Phi = 2\phi + \lambda + 5\nu, \quad B = C_\mu dx^\mu \wedge dx^4,$$

where greek indices run over the four non-compact space-time dimensions. At infinity the radius of the S^1 circle is R and the volume of the five torus is V_5 . We shall fix the asymptotic value of the scalar fields λ and ν to be zero. Then, the reduced effective action has the form

$$\frac{1}{16\pi G_N} \int d^4x \sqrt{-G} e^{-2\phi} \left[R + 4(\partial\phi)^2 - (\partial\lambda)^2 - 5(\partial\nu)^2 - \frac{e^{2\lambda}}{4} (dA)^2 - \frac{e^{-2\lambda}}{4} (dC)^2 \right].$$

The four-dimensional Newton constant is now given by

$$8G_N = g^2 \alpha' ,$$

where g is the four-dimensional string coupling defined by

$$g^2 = g_s^2 \frac{(2\pi)^6 \alpha'^3}{2\pi R V_5} .$$

The field ν will only be important for us to study fluctuations of the black hole geometry, so for now we set $\nu = 0$.

The above action admits the BPS black hole solution [4]

$$\begin{aligned} ds^2 &= -(FH)^{-1} dt^2 + ds^2(\mathbb{E}^3) , \\ e^{2\phi} &= (FH)^{-1/2} , \quad e^\lambda = \left(\frac{F}{H}\right)^{1/2} , \\ A &= F^{-1} dt , \quad C = H^{-1} dt , \end{aligned} \tag{2.1}$$

where F and H are harmonic functions on \mathbb{E}^3 . For single-centered black holes we have

$$F = 1 + \frac{\rho_n}{\rho} , \quad H = 1 + \frac{\rho_m}{\rho} ,$$

where ρ_n and ρ_m are constants and ρ is the radial coordinate on \mathbb{E}^3 .

To quantize the black hole charges one can compute its ADM mass and equate it to the mass of a string with momentum and winding (n, m) . Moving to the Einstein metric $e^{-2\phi} G_{\mu\nu}$ we can compute the ADM mass

$$M = \frac{1}{4G_N} (\rho_n + \rho_m) = \frac{n}{R} + \frac{mR}{\alpha'} .$$

Thus, the constants ρ_n and ρ_m are given by

$$\rho_n = \frac{g^2 \alpha'}{2} \frac{n}{R} , \quad \rho_m = \frac{g^2}{2} mR .$$

The classical limit, where the gravity description is adequate, consists in letting $g \rightarrow 0$ and $n, m \rightarrow \infty$ with the charges ρ_n, ρ_m kept finite. As we shall see below, *within this limit*, one needs to include curvature corrections to the effective gravity action.

Let us look at the region close to the black hole $\rho \ll \rho_n, \rho_m$. The string metric and remaining background fields become

$$\begin{aligned} ds^2 &= -\frac{\rho^2}{\rho_n \rho_m} dt^2 + d\rho^2 + \rho^2 d\Omega^2 , \\ e^{2\phi} &= \frac{\rho}{\sqrt{\rho_n \rho_m}} , \quad e^\lambda = \sqrt{\frac{\rho_n}{\rho_m}} , \\ A &= \frac{\rho}{\rho_n} dt , \quad C = \frac{\rho}{\rho_m} dt . \end{aligned} \tag{2.2}$$

In the limit $\rho \rightarrow 0$ the geometry is singular, but the string coupling goes to zero. For fixed angles on the 2-sphere the metric is just the Rindler metric so that $\rho = 0$ is a singular horizon. Since the 2-sphere has radius ρ , the Ricci scalar will diverge as ρ^{-2} and string corrections will be important close to the horizon. Nevertheless, when $\sqrt{\alpha'} \ll \rho_n, \rho_m$, the above form of the fields (2.2) can be trusted in the region $\sqrt{\alpha'} \ll \rho \ll \rho_n, \rho_m$.

To study the near-horizon geometry one needs to consider all α' corrections to the tree level supergravity action. It turns out that, including a supersymmetric completion of curvature square terms in the action, the near horizon geometry is given by $AdS_2 \times S^2$ [5]. The corresponding Wald entropy formula [6] exactly reproduces the degeneracy of BPS string states with the same charges [7]. We remark that in this scheme the field λ diverges at the horizon, but it is not the field associated to the radius of S^1 [12]. The same geometry can be obtained by including only the Gauss-Bonnet correction in the Heterotic effective action [19]. In this case, although the effective action is incomplete, the scalar field λ is fixed at the horizon and the Wald entropy still reproduces the string degeneracy. There is now strong evidence that the α' corrected near-horizon solution is

$$\begin{aligned}
 ds^2 &= -\frac{2\rho^2}{\rho_n\rho_m} dt^2 + \frac{\alpha'}{2\rho^2} d\rho^2 + \frac{\alpha'}{2} d\Omega^2, \\
 e^{2\phi} &= \sqrt{\frac{\alpha'}{\rho_n\rho_m}}, & e^\lambda &= \sqrt{\frac{\rho_n}{\rho_m}}, \\
 A &= \frac{2\rho}{\rho_n} dt, & C &= \frac{2\rho}{\rho_m} dt.
 \end{aligned}
 \tag{2.3}$$

The AdS_2 and S^2 radii squared are both $\alpha'/2$. At the horizon, the square of the four-dimensional string coupling

$$g^2 e^{2\phi} = \frac{2}{\sqrt{nm}}$$

is completely fixed by the charges (n, m) . Thus, in the classical limit, loop corrections are negligible throughout the whole space time. The other scalar field λ is also fixed, with the radius of the S^1 circle at the horizon given by

$$Re^\lambda = \sqrt{\alpha' \frac{n}{m}}.$$

Finally, the area of the horizon in the Einstein frame is

$$A_h = \pi\alpha' g^2 \sqrt{nm} \simeq 2G_N S.$$

Although the Wald entropy for the above near-horizon geometry exactly reproduces the string degeneracy, we can not be completely certain of the precise numerical factors in the solution because higher α' -corections may change them. In the next section we shall assume this specific form of the metric in the cross section computations.

Sen [12] gave strong evidence that an interpolating solution between the near-horizon geometry (2.3) and the geometry close to the black hole (2.2) exists provided $\sqrt{\alpha'} \ll \rho_n, \rho_m$. For this range of parameters one has

$$g^2 \sqrt{nm} \gg 1.$$

As described in the Introduction and in section 3, increasing the coupling to $g^2\sqrt{nm} \sim 1$ the average size of the string states shrinks to $\sqrt{\alpha'}$. The string has then the same size of the above horizon, therefore forming a black hole. This supergravity solution describes the final stage of the transition.

For $\rho_n, \rho_m \ll \sqrt{\alpha'}$, a solution interpolating between the $AdS_2 \times S^2$ near-horizon geometry and the asymptotically flat geometry (2.1) is not known.² This range of parameters corresponds to the weakly coupled string phase

$$g^2\sqrt{nm} \ll 1,$$

where the size of string states is much larger than the radius of the AdS horizon. Therefore we believe that such an interpolating solution does not exist. This is a very interesting open problem on which we shall comment in the Conclusion.

2.2 Absorption cross section

We shall study the absorption cross section of massless fields by the Heterotic black hole just described. In particular, we concentrate our attention on minimally coupled scalars, like the fluctuation ν of the volume of the internal T^5 , and on fixed scalars, like the field λ associated to the S^1 radius.

2.2.1 Equations of motion

Near infinity, where the curvature is small and α' corrections are negligible, the equations of motion for the field fluctuations can be conveniently derived from the effective 4-dimensional action in the Einstein frame

$$\frac{1}{16\pi G_N} \int d^4x \sqrt{-G} \left[R - 2(\partial\phi)^2 - (\partial\lambda)^2 - 5(\partial\nu)^2 - \frac{e^{-2\phi+2\lambda}}{4} (dA)^2 - \frac{e^{-2\phi-2\lambda}}{4} (dC)^2 \right].$$

Thus, far from the black hole, the field ν is minimally coupled and obeys the massless Klein-Gordon equation $\square\nu = 0$. In general, however, the equations of motion change due to α' corrections, even though they will still be invariant under rescaling of the T^5 and S^1 volumes, T-duality transformations and shifts of the dilaton field. More precisely, the corrected effective action is invariant under the set of transformations

$$\begin{aligned} \nu &\rightarrow \nu + \alpha; \\ \lambda &\rightarrow \lambda + \beta, \quad dA \rightarrow e^{-\beta} dA, \quad dC \rightarrow e^{\beta} dC; \end{aligned} \tag{2.4}$$

$$\begin{aligned} \nu &\rightarrow -\nu; \\ \lambda &\rightarrow -\lambda, \quad dA \leftrightarrow dC; \end{aligned} \tag{2.5}$$

$$\phi \rightarrow \phi + \gamma, \quad G_N \rightarrow e^{-2\gamma} G_N; \tag{2.6}$$

²Interpolating solutions between AdS and infinity have been studied in [12, 13]. In both papers these solutions present oscillations which were considered unphysical in [12]. It was also argued that these oscillations could be removed by field redefinitions. Using the same setup we numerically found interpolating solutions for all values of the charges. However, while for large charges, $\sqrt{\alpha'} \ll \rho_n, \rho_m$, the solution has small oscillations around the physical background, for small charges, $\rho_n, \rho_m \ll \sqrt{\alpha'}$, the oscillations dominate the asymptotic behaviour of the field.

with α, β constants. The cross section computations here performed rely on a field representation such that the effective action has the above symmetries explicitly realized. We shall assume that the geometry (2.3) is a solution of such an effective action.

Let us first focus on the field ν . The effective action will depend only on derivatives of the field and will be even in ν . This fact implies that fluctuations of ν do not mix with other fluctuations. Since the $AdS_2 \times S^2$ geometry is a product of maximally symmetric spaces, the near-horizon background fields respect the symmetries of this space. Therefore, the only tensors available to define a differential equation near the horizon are the metric and volume forms. Focusing on spherically symmetric perturbations, the equation for the scalar field ν must be of the form

$$f(\alpha' \square) \nu = 0, \tag{2.7}$$

where the function $f(x) = f_1 x + f_2 x^2 + \dots$ depends, in principle, on the hole charges. Notice that (2.7) has no mass term due to the symmetry $\nu \rightarrow \nu + \alpha$. Hence solutions of the Klein-Gordon equation $\square \nu = 0$ are still solutions of the generic equation (2.7) and we expect those to be the physical solutions. Possible other solutions to (2.7) should be removed after field redefinitions. We conclude that the near-horizon action for s-wave physical fluctuations of the field ν has the generic form

$$\frac{5}{16\pi G_N} \int d^4x \sqrt{-G} e^{-2\phi} K^{\alpha\beta} \partial_\alpha \nu \partial_\beta \nu, \tag{2.8}$$

written in the string frame, and where $2K_{\alpha\beta} = c G_{\alpha\beta}$. We now argue that the proportionality constant c is of order unity and independent of the hole charges. In fact, rescaling the time coordinate and using the symmetries (2.4) and (2.6) one can remove the dependence of the near-horizon solution (2.3) on ρ_n and ρ_m [8]. In this new solution the constant c must be independent of the hole charges. This result holds in the original solution because both the metric and $K_{\alpha\beta}$ are invariant under symmetries (2.4) and (2.6).

The study of fluctuations of the fixed scalars λ and ϕ and of the metric is more subtle since these fields are in general coupled to each other. However, if the two charges ρ_n and ρ_m are equal, the S^1 circle can have a constant background radius and the s-wave fluctuations of the field λ decouple from other fluctuations [30]. Let us first recall that, in general, the equations of motion for the gauge fields A, C can always be put into the form $d\mathcal{F} = d\mathcal{H} = 0$, where the two-forms \mathcal{F}, \mathcal{H} are given, to leading order in α' , by

$$\mathcal{F} \simeq e^{2\lambda-2\phi} \star dA, \quad \mathcal{H} \simeq e^{-2\lambda-2\phi} \star dC,$$

and transform under (2.4) as $\mathcal{F} \rightarrow e^\beta \mathcal{F}, \mathcal{H} \rightarrow e^{-\beta} \mathcal{H}$. For spherically symmetric solutions (and their fluctuations), the forms \mathcal{F}, \mathcal{H} are constant along the transverse sphere

$$\mathcal{F} = \rho_n \epsilon(S^2), \quad \mathcal{H} = \rho_m \epsilon(S^2),$$

with $\epsilon(S^2)$ the volume form of S^2 . Furthermore, their value is fixed by the overall charges ρ_n, ρ_m of the configuration. In particular, we will study fluctuations around a specific extremal solution with

$$\lambda = 0, \quad A = C,$$

which is invariant under the T-duality transformation (2.5), so that $\rho_n = \rho_m$. We will denote with a the odd fluctuation of the gauge fields $\delta A = a$, $\delta C = -a$. The fields λ and a are the only fluctuations which are odd under (2.5) and this implies that their linear terms mix only among each other. Therefore, the relevant quadratic part of the action reads

$$\frac{1}{16\pi G_N} \int \left(\frac{1}{2} \lambda \mathcal{D}_4 \lambda + da \wedge \mathcal{D}_2 \lambda - \frac{1}{2} da \wedge \mathcal{D}_0 da \right),$$

where the \mathcal{D}_i are differential operators whose leading terms in the α' expansions are given by

$$\begin{aligned} \mathcal{D}_4 &\simeq 2d \star d - 4e^{-2\phi} dA \wedge \star dA, \\ \mathcal{D}_2 &\simeq -4e^{-2\phi} (\star dA) \wedge, \\ \mathcal{D}_0 &\simeq 2e^{-2\phi} \star. \end{aligned}$$

The equation of motion for a is given by $d(\mathcal{D}_0 da - \mathcal{D}_2 \lambda) = 0$. As discussed above, for spherically symmetric fluctuations this equation is solved by

$$\mathcal{D}_0 da - \mathcal{D}_2 \lambda = \delta \epsilon (S^2),$$

where $\delta = \delta \rho_n - \delta \rho_m$ is the change in the charge of the solution. We shall focus on perturbations which do not change the charges so that $\delta = 0$. Inverting the operator \mathcal{D}_0 in power series in α' we may solve for da as

$$da = \frac{1}{\mathcal{D}_0} \mathcal{D}_2 \lambda.$$

Defining the operator \mathcal{D}_2^\dagger by $\int da \wedge \mathcal{D}_2 \lambda = \int \lambda \wedge \mathcal{D}_2^\dagger da$, so that

$$\mathcal{D}_2^\dagger \simeq -4e^{-2\phi} (\star dA) \wedge,$$

the equation of motion for λ reads $\mathcal{D}_4 \lambda + \mathcal{D}_2^\dagger da = 0$ and can be recasted in the final form

$$\mathcal{D} \lambda = 0,$$

with $\mathcal{D} = \mathcal{D}_4 + \mathcal{D}_2^\dagger \frac{1}{\mathcal{D}_0} \mathcal{D}_2$. To leading order in α' the fluctuation equation for λ reads

$$\square \lambda \simeq 2\rho_n^2 \frac{e^{2\phi}}{r^4} \lambda, \tag{2.9}$$

where r is the radius of the S^2 in the Einstein frame which depends on the radial coordinate ρ . In the near-horizon region, using the same symmetry arguments as for the minimally coupled scalar, the operator \mathcal{D} has the generic form $\star \mathcal{D} = g(\alpha' \square)$ with $g(x) = g_0 + g_1 x + \dots$. We then expect the physical solutions to satisfy

$$\square \lambda = \mu^2 \lambda, \tag{2.10}$$

where the mass μ is defined by the relation $g(\alpha' \mu^2) = 0$.

Now we argue that the mass μ is non-vanishing. Due to the attractor mechanism, there must be a family of extremal solutions with the same charges $\rho_n = \rho_m$ and with $\lambda \simeq \lambda_\infty$ arbitrary at infinity and fixed at the horizon with $\lambda_h = 0$. To leading order in α' , this family is given by (2.1) with the harmonic functions F, H now given by

$$F = e^{\lambda_\infty} + \frac{\rho_n}{\rho}, \quad H = e^{-\lambda_\infty} + \frac{\rho_m}{\rho}.$$

For small λ_∞ this family of solutions generates fluctuations of the form studied above. Therefore, in the near-horizon region, we expect that the regular static solution to (2.10) actually vanishes for $\rho \rightarrow 0$. This is only possible if $\mu^2 \neq 0$, since in this case the regular solution is exponentially damped at the horizon. In fact, when $\mu^2 = 0$ the regular solution is constant and cannot describe a solution attracted to $\lambda = 0$ at $\rho \rightarrow 0$. In the remainder of this paper we shall leave the α' corrected mass μ as a parameter since its computation is beyond our reach.³

We shall see that, for small frequencies, the form of the equations of motion far from the black hole and near its horizon will be enough to determine the energy dependence of the cross section.

2.2.2 Cross section

We will now analyze the absorption cross section for the s-wave scalar fluctuation considered in the last section, in the limit of vanishing frequency $\omega \rightarrow 0$ of the incoming wave. As it is well known, if we consider the equations of motion only to leading order in α' , the propagation of fluctuations of fixed energy and angular dependence can be rewritten as an effective one-dimensional Schrödinger problem. Cross sections are then simply related to transmission probabilities in the presence of an effective potential. In the previous section we have shown that, to all orders in α' but only in the asymptotic regions far from the black hole and near its horizon, the linearized equations for some specific s-wave scalar fluctuation have the general form

$$\square\Psi = U(\rho)\Psi$$

and are therefore rewritable as effective Schrödinger problems. This asymptotic form of the equations of motion will nonetheless suffice to determine the energy dependence of the cross section in the limit $\omega \rightarrow 0$. Note that we shall *not* assume that the equations of motion can be recasted in the Schrödinger form in the intermediate region which interpolates between the horizon and Minkowski infinity. The case of the minimally coupled field ν corresponds in general to $U = 0$. When considering the fixed scalar λ we will, on the other hand, concentrate on the particular background with $\rho_n = \rho_m$ and $\lambda = 0$, $A = C$.

To reduce the equations of motion in the asymptotic regions to a one-dimensional Schrödinger problem, we write the Einstein metric in the form

$$ds_E^2 = ds_2^2 + r^2(x) d\Omega_2^2.$$

³One could naively use the uncorrected form of the equation of motion (2.9) to find $\mu^2 = 8/(\rho_n\sqrt{\alpha'})$. This mass, however, must be corrected with the inclusion of all the higher α' terms.

We have switched radial variable from ρ to the usual Tortoise coordinate $x = x(\rho)$, which puts the two-dimensional metric in the conformally flat form

$$ds_2^2 = a^2(x) (-dt^2 + dx^2) .$$

The horizon is at $x \rightarrow -\infty$ and Minkowski infinity at $x \rightarrow +\infty$. Decomposing the field as

$$\Psi(t, x, \theta, \varphi) = \frac{e^{i\omega t}}{r(x)} \psi(x)$$

the asymptotic field equations are in the Schrödinger form

$$\left(\omega^2 + \frac{d^2}{dx^2} \right) \psi(x) = V(x) \psi(x) ,$$

with an effective potential

$$V = a^2 U + \frac{1}{r} \frac{d^2}{dx^2} r .$$

In appendix A we show that the leading frequency dependence for $\omega \rightarrow 0$ of the transmission probability $|T|^2$ is determined only by the asymptotic behaviour of the potential, irrespective of the form of the equations of motion in the intermediate region. A potential tail for $x \rightarrow \pm\infty$ such that $x^2 V(x)$ tends to

$$0, \quad \beta^2 - 1/4, \quad +\infty$$

contributes, respectively, to the transmission probability with a factor of

$$\omega, \quad \omega^{2\beta}, \quad \exp \left(\mp 2 \int^{x_{\pm}(\omega)} \sqrt{V(x)} dx \right) ,$$

where $x_{\pm}(\omega)$ are the classical turning points, which tend to $\pm\infty$ as $\omega \rightarrow 0$. The numerical coefficient in front of this leading frequency dependence is determined from the exact solution of the $\omega = 0$ equation with finite amplitude at the horizon (see appendix A).

Let us describe in detail the case of the minimally coupled scalar ν . The following derivation follows closely [31] but shows that the result can be obtained without the knowledge of the equations of motion in the intermediate region. We start with a purely infalling wave at the horizon

$$\psi(x) \simeq e^{i\omega x}, \quad x \rightarrow -\infty .$$

Since the effective potential at the horizon decays faster than $1/x^2$, the asymptotic wave length grows faster than $|x_-|$ as ω tends to zero. Therefore, the wave function will penetrate far beyond the turning point keeping its profile, i.e.

$$\psi(x) \simeq 1 + i\omega x, \quad x_- \ll x \ll \epsilon x_-, \quad (2.11)$$

where $\epsilon \ll 1$. In the region $x_- \ll x \ll x_+$ we have $\omega^2 \ll V(x)$ so that we can drop the ω^2 term in Schrödinger equation and consider the solution at zero frequency. In fact,

the equations of motion need not even be of the Schrödinger form in the intermediate region. We only need the solution at $\omega = 0$ which connects to the asymptotic form of the field (2.11). Moreover, since we are interested in the $\omega \rightarrow 0$ limit, we only need to evolve the constant term 1. For this purpose, we use the constant solution for ν which is valid to all orders in α' . At infinity, the effective action for s-wave perturbations of the field ν takes the form

$$S \simeq \frac{5}{4G_N} \int dt dx \left[(\partial_t \psi)^2 - (\partial_x \psi)^2 - \frac{\partial_x^2 r}{r} \psi^2 \right],$$

with $\psi(t, x) = \nu(t, x) r(x)$. As explained above, symmetry arguments fix the effective action in the near-horizon region to have the form (2.8), yielding

$$S \simeq \frac{5}{4G_N} \int dt dx \frac{c}{2} R_h^2 \left[(\partial_t \nu)^2 - (\partial_x \nu)^2 \right],$$

where R_h is the horizon radius in the Einstein frame and $c/2$ is the proportionality constant between the effective near-horizon metric $K_{\alpha\beta}$ seen by the scalar field and the string metric. Thus, the exact solution $\nu = \sqrt{2}/(\sqrt{c}R_h)$ corresponds to $\psi = 1$ at the horizon and to $\psi \simeq \sqrt{2}x/(\sqrt{c}R_h)$ at infinity. We can then evolve the term 1 in equation (2.11), obtaining

$$\psi(x) \simeq \sqrt{\frac{2}{c}} \frac{x}{R_h}, \quad \epsilon x_+ \ll x \ll x_+,$$

and, since once again one has a fast decaying potential at infinity,

$$\psi(x) \simeq \frac{1}{i\sqrt{2c\omega R_h}} (e^{i\omega x} - e^{-i\omega x}), \quad x \rightarrow \infty.$$

Hence, the transmission probability for the s-wave is

$$|T_0|^2 \simeq 2cR_h^2 \omega^2.$$

Moreover, since the total cross section is given by

$$\sigma_{abs} = \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l+1) |T_l|^2,$$

and for small frequencies only the s-wave contributes, we obtain

$$\sigma_{abs} \simeq \frac{c}{2} A_h \simeq c G_N S.$$

The result $\sigma_{abs} \simeq A_h$ is generic for fields obeying the massless Klein-Gordon equation throughout the whole space [31]. Here we have generalized it to the case where we only know the interpolating constant solution, together with the fact that the field obeys the massless Klein-Gordon equation in both asymptotic regions.

Let us now consider the case of the fixed scalar λ . At infinity the potential decays as $1/x^3$, while near the horizon it goes as

$$V \simeq \frac{\mu^2 R_h^2}{x^2},$$

where μ is the unknown mass term. In this case the small frequency behaviour of the absorption cross section reads

$$\sigma_{abs} \sim \omega^{2\beta-1},$$

where

$$2\beta = \sqrt{1 + (2\mu R_h)^2}.$$

As explained in the previous section, *the attractor mechanism implies that $\mu > 0$* , so that the absorption cross section for the fixed scalar vanishes in the $\omega \rightarrow 0$ limit.⁴ On the other hand, we cannot fix the normalization since we do not have an explicit zero-frequency solution valid in the entire intermediate region. The absence of absorption at zero frequency comes as no surprise since this scalar has its value at the horizon fixed by the hole charges. This suppresses fluctuations close to the horizon and therefore the associated infalling flux.

3. String phase

In this section we use perturbative string theory to study the string states corresponding to the black hole of the previous section and to compute their absorption cross section. Let X_L^μ ($\mu = 0, \dots, 25$) and $X_R^\mu, \tilde{\psi}^\mu$ ($\mu = 0, \dots, 9$) be, respectively, the left- and right-moving worldsheet matter fields of the Heterotic string, where directions $\mu = 0, \dots, 3$ correspond to the four-dimensional spacetime, directions $\mu = 4, 5, \dots, 9$ are compactified on $S^1 \times T^5$ and the remaining left moving directions $\mu = 10, \dots, 25$ correspond to the internal compact Heterotic torus. The matter Virasoro constraints read $L_0 = \tilde{L}_0 = 0$, with

$$L_0 = N + \frac{\alpha'}{4} \mathbf{p}_L^2 - 1,$$

$$\tilde{L}_0 = \tilde{N} + \frac{\alpha'}{4} \mathbf{p}_R^2 - \frac{1}{2} \delta_{\text{NS}},$$

where $\delta_{\text{NS}} = 1$ in the NS sector and vanishes in the R sector. The N 's and \mathbf{p} 's are the usual left- and right-moving levels and momenta.

3.1 String states

The black hole is a BPS massive state of the Heterotic theory compactified on $S^1 \times T^5$. The corresponding string has Kaluza-Klein momentum and winding (n, m) along the S^1 direction, such that

$$p_{L,R}^4 = \left(\frac{n}{R} \mp \frac{mR}{\alpha'} \right).$$

The T^5 directions are mere spectators and we shall consider black holes which are chargeless under the internal gauge group, so that $p_{L,R}^\mu = 0$ for $\mu \geq 5$. Therefore, the string has total left/right momentum

$$\mathbf{p}_{L,R} = (p, p_{L,R}^4, 0).$$

⁴If one naively neglects the α' corrections to the mass μ (see previous footnote) the cross section satisfies $\sigma_{abs} \sim \omega^{\sqrt{17}-1} \sim \omega^{3.12}$.

In this notation, the unbold momentum p is four-dimensional. The BPS condition

$$\mathbf{p}_R \cdot \mathbf{p}_R = 0$$

fixes the four-dimensional mass to be

$$M^2 = -p \cdot p = \left(\frac{n}{R} + \frac{mR}{\alpha'} \right)^2 .$$

The Virasoro constraints $L_0 = \tilde{L}_0 = 0$ then imply

$$N = nm + 1, \quad \tilde{N} = \frac{1}{2} \delta_{\text{NS}} .$$

There are many possible ways of satisfying these conditions corresponding to different black hole microstates. In the string phase we shall average over all the string microstates.

In the NS sector the string states are space-time bosons with vertex operator in the -1 picture given by

$$\mathcal{V}_i(z, \bar{z}) = c \tilde{c} e^{-\tilde{\phi}} \mathcal{V}_i(z) \zeta \cdot \tilde{\psi} e^{i \mathbf{p}_R \cdot \mathbf{X}_R}, \quad (3.1)$$

where the holomorphic matter part $\mathcal{V}_i(z)$ is a conformal primary with $L_0 = 0$ and momentum \mathbf{p}_L . The string polarization vector ζ satisfies the physical condition

$$\zeta \cdot \mathbf{p}_R = 0,$$

with normalization $\zeta \cdot \zeta = 1$.

In the R sector the string states are space-time fermions with vertex operator in the $-1/2$ picture given by

$$\mathcal{V}_i(z, \bar{z}) = c \tilde{c} e^{-\tilde{\phi}/2} \mathcal{V}_i(z) u^\alpha \tilde{\Theta}_\alpha e^{i \mathbf{p}_R \cdot \mathbf{X}_R}, \quad (3.2)$$

where u^α is a chiral spinor and $\tilde{\Theta}_\alpha$ is the R state spin field. Physical states obey the Dirac equation $\not{p}u = 0$ and are normalized by $\bar{u}\Gamma^0 u = 2p^0$.

For fixed charges (n, m) there are many possible left-moving string microstates which are conveniently parametrized using DDF operators [32]. To be specific, we work in the center-of-mass frame, so that the four-dimensional momentum p is given by

$$p = (M, 0, 0, 0) .$$

Choosing \mathbf{k} to be the null vector with $k^\mu = 0$ for $\mu \geq 4$ and

$$k = -\frac{2}{M\alpha'} (1, 1, 0, 0) ,$$

we define the DDF operators

$$A_n^i \equiv \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i} i \partial X_L^i e^{i n \mathbf{k} \cdot \mathbf{X}_L(z)},$$

with n integer and X^i ($i = 2, \dots, 25$) one of the 24 directions transverse to \mathbf{k} . These operators obey

$$[L_m, A_n^i] = 0, \quad [A_n^i, A_m^j] = n \delta^{ij} \delta_{m+n},$$

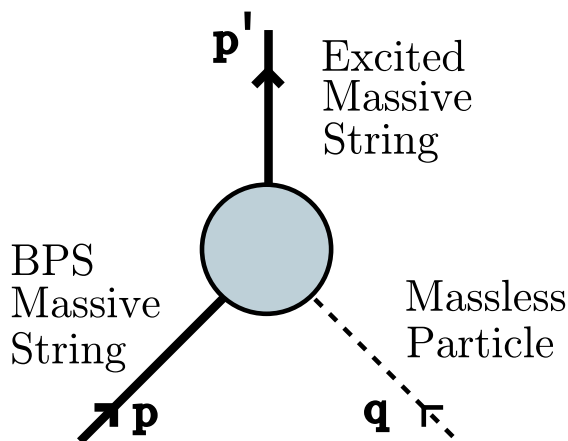


Figure 3: A BPS massive string state absorbs a massless particle becoming a low lying excited string state above the BPS bound.

when acting on states whose momentum \mathbf{p}_L satisfies $\alpha' \mathbf{p}_L \cdot \mathbf{k} = 2$. Then, the normalized holomorphic matter part of the vertex operator can be written as

$$\mathcal{V}_i(z) = \prod_{i=2}^{25} \prod_{n=1}^{\infty} \frac{1}{\sqrt{r_n^i n^{r_n^i}}} (A_{-n}^i)^{r_n^i} e^{i(\mathbf{p}_L + N\mathbf{k}) \cdot \mathbf{X}_L(z)},$$

where the r_n^i are occupation numbers satisfying $\sum_{i=2}^{25} \sum_{n=1}^{\infty} n r_n^i = N$. In the operator language the string states are constructed by applying DDF operators to the left-moving tachyonic ground state with momentum $\mathbf{p}_L + N\mathbf{k}$. The string left-moving degeneracy is then given by the number of partitions $\{r_n^i\}$ of the level N in integers of 24 colours, corresponding to the 24 transverse directions.

3.2 Absorption amplitude

In this section we will determine the absorption cross section of a massless field by the massive BPS string states just described as represented in figure 3. We compute first the absorption amplitude of a massless boson by bosonic microstates of the BPS massive string. The incoming massive state has vertex operator given by (3.1). The external massless particle is described by the vertex operator in the 0 picture

$$\mathcal{V}(z, \bar{z}) = \frac{2}{\alpha'} c \bar{c} i \partial X_L^l \left(i \bar{\partial} X_R^j + \frac{\alpha'}{2} \mathbf{q} \cdot \tilde{\psi} \tilde{\psi}^j \right) e^{i\mathbf{q} \cdot (\mathbf{X}_L + \mathbf{X}_R)},$$

where \mathbf{q} is the particle's momentum, with the only non-vanishing components given by $q^0 = q^1 = \omega$, where ω is the energy in the frame of the BPS string state. The transverse polarization directions l and j then run from 2 to 25 and from 2 to 10, respectively. Note that we have chosen, for reasons that will become clear soon, the momentum \mathbf{k} used to define the DDF operators to be collinear with the momentum \mathbf{q} , so that

$$\mathbf{q} = -\varepsilon \mathbf{k}, \quad \varepsilon = \frac{\omega M \alpha'}{2}.$$

The excited massive string final state will have momentum $\mathbf{p}'_{L,R} = \mathbf{p}_{L,R} + \mathbf{q}$ and vertex operator of the form

$$\mathcal{V}_f(z, \bar{z}) = c \tilde{c} e^{-\tilde{\phi}} \mathcal{V}_f(z) \tilde{\mathcal{V}}_f(\bar{z}),$$

where $\mathcal{V}_f(z)$ and $\tilde{\mathcal{V}}_f(\bar{z})$ are conformal primaries of the matter theory with $L_0 = \tilde{L}_0 = 0$.

The tree level S -matrix element for this process is given by

$$\frac{4\kappa}{\alpha'} \langle \mathcal{V}_f^*(z_3, \bar{z}_3) \mathcal{V}(z_2, \bar{z}_2) \mathcal{V}_i(z_1, \bar{z}_1) \rangle = i (2\pi)^4 \delta^{(4)}(p' - p - q) \mathcal{M}.$$

The normalization is given by the four-dimensional gravitational coupling κ defined as usual by

$$2\kappa^2 = 16\pi G_N = 2\pi g^2 \alpha'$$

and is fixed, for example, by matching the three graviton coupling with the one coming from the low energy effective action. The matrix element \mathcal{M} is then given by

$$\mathcal{M} = \frac{4\kappa}{\alpha'} C \tilde{C}, \tag{3.3}$$

where the dimensionless constants C and \tilde{C} come, respectively, from the holomorphic and anti-holomorphic parts of the correlation function

$$C = \sqrt{\frac{2}{\alpha'}} z_{12} z_{23} z_{13} \left\langle \mathcal{V}_f^*(z_3) i\partial X_L^l e^{i\mathbf{q}\cdot\mathbf{X}_L(z_2)} \mathcal{V}_i(z_1) \right\rangle,$$

$$\tilde{C} = \sqrt{\frac{2}{\alpha'}} \bar{z}_{12} \bar{z}_{23} \left\langle \tilde{\mathcal{V}}_f^*(\bar{z}_3) \left(i\bar{\partial} X_R^j + \frac{\alpha'}{2} \mathbf{q} \cdot \tilde{\boldsymbol{\psi}} \tilde{\psi}^j \right) e^{i\mathbf{q}\cdot\mathbf{X}_R(\bar{z}_2)} \boldsymbol{\zeta} \cdot \tilde{\boldsymbol{\psi}} e^{i\mathbf{p}\cdot\mathbf{X}_R(\bar{z}_1)} \right\rangle,$$

where $z_{ij} = z_i - z_j$. The z_i dependence of the matter 3-pt functions is fixed by conformal invariance and precisely cancels the explicit factors of z_{ij} coming from the ghosts. Thus, one can take the limit $z_2 \rightarrow z_1$ and replace the BPS massive string and massless particle vertex operators by their OPE. Non-singular terms of the OPE disappear in the $z_2 \rightarrow z_1$ limit because of the explicit factors of z_{12} and \bar{z}_{12} . On the other hand, terms more singular than z_{12}^{-1} can not contribute because they would give a divergent answer. Hence, the only contribution comes from the z_{12}^{-1} term in the OPE, which is given by the contour integral of the massless particle vertex around the massive string one. The result is a conformal primary with the same weight of the massive string vertex because the contour integral of weight 1 holomorphic primaries defines an operator that commutes with the Virasoro generators. One is then left with a two-point function, so that this conformal primary is, up to normalization, the final excited string state

$$C \mathcal{V}_f(z_1) = \sqrt{\frac{2}{\alpha'}} \oint_{z_1} \frac{dz_2}{2\pi i} i\partial X_L^l e^{i\mathbf{q}\cdot\mathbf{X}_L(z_2)} \mathcal{V}_i(z_1),$$

$$\tilde{C} \tilde{\mathcal{V}}_f(\bar{z}_1) = \sqrt{\frac{2}{\alpha'}} \oint_{\bar{z}_1} \frac{d\bar{z}_2}{2\pi i} \left(i\bar{\partial} X_R^j + \frac{\alpha'}{2} \mathbf{q} \cdot \tilde{\boldsymbol{\psi}} \tilde{\psi}^j \right) e^{i\mathbf{q}\cdot\mathbf{X}_R(\bar{z}_2)} \boldsymbol{\zeta} \cdot \tilde{\boldsymbol{\psi}} e^{i\mathbf{p}\cdot\mathbf{X}_R(\bar{z}_1)}.$$

The only quantities that remain to be computed are the above normalization constants C and \tilde{C} . Let us start by computing C . The computation greatly simplifies since we have

chosen \mathbf{k} , appearing in the DDF operators, to be collinear with the momentum $\mathbf{q} = -\varepsilon\mathbf{k}$ of the massless particle. Using $\alpha'\mathbf{p} \cdot \mathbf{k} = 2$, it is easy to show that $\varepsilon = \omega M\alpha'/2$ is related to the square of the center-of-mass energy $s = -(p+q)^2$ through

$$\varepsilon = \frac{\alpha'}{4} (s - M^2) .$$

The absorption process can only occur if $s = M'^2$, for some excited level of the Heterotic string with mass M' and the same charges n and m . Therefore,

$$\varepsilon = \frac{\alpha'}{4} (M'^2 - M^2) \tag{3.4}$$

is bound to be an integer by the L_0 and \tilde{L}_0 physical conditions

$$\varepsilon = \tilde{N}' - \frac{1}{2} \delta_{\text{NS}} = N' - N ,$$

where N' and \tilde{N}' are the levels of the final state. Finally, using the DDF operator parametrization of the string microstate $\mathcal{V}_i(z)$ one obtains

$$C \mathcal{V}_f(z) = A_{-\varepsilon}^l \prod_{i=2}^{25} \prod_{n=1}^{\infty} \frac{1}{\sqrt{r_n^i! n^{r_n^i}}} (A_{-n}^i)^{r_n^i} e^{i(\mathbf{p}_L + N\mathbf{k}) \cdot \mathbf{X}_L(z)} ,$$

which fixes

$$C = \sqrt{\varepsilon(r_\varepsilon^l + 1)} .$$

To determine the other constant \tilde{C} we define the DDF operator

$$\tilde{A}_{-\varepsilon}^j \equiv \sqrt{\frac{2}{\alpha'}} \oint \frac{d\bar{z}}{2\pi i} \left(i \bar{\partial} X_R^j + \frac{\alpha'}{2} \mathbf{q} \cdot \tilde{\psi} \tilde{\psi}^j \right) e^{i\mathbf{q} \cdot \mathbf{X}_R(\bar{z})} ,$$

for $\mathbf{q} = -\varepsilon\mathbf{k}$ and integral ε , and write

$$\tilde{C} |\tilde{\mathcal{V}}_f\rangle = \tilde{A}_{-\varepsilon}^j |\tilde{\mathcal{V}}_i\rangle , \quad \tilde{C}^2 = \langle \tilde{\mathcal{V}}_i | \tilde{A}_\varepsilon^j \tilde{A}_{-\varepsilon}^j | \tilde{\mathcal{V}}_i \rangle ,$$

where $|\tilde{\mathcal{V}}_i\rangle$ denotes the right-moving matter part of the initial string state. Then, since $[\tilde{A}_\varepsilon^j, \tilde{A}_{-\varepsilon}^j] = \varepsilon$ and since $\tilde{A}_\varepsilon^j |\tilde{\mathcal{V}}_i\rangle = 0$ for $\varepsilon \geq 1$, we arrive at the simple result

$$\tilde{C} = \sqrt{\varepsilon} .$$

We have then computed the absorption matrix element (3.3)

$$\mathcal{M} = \frac{4\kappa}{\alpha'} \varepsilon \sqrt{r_\varepsilon^l + 1} . \tag{3.5}$$

3.2.1 Fermionic states

The computation of the absorption amplitude of massless bosons by fermionic microstates of the BPS massive string proceeds along similar lines, with the same result (3.5) for the absorption amplitude. The initial string vertex operator is given by (3.2) and the final one can be taken to be $\mathcal{V}_f(z, \bar{z}) = c \tilde{c} e^{-\tilde{\phi}/2} \mathcal{V}_f(z) \tilde{\mathcal{V}}_f(\bar{z})$, where $\mathcal{V}_f(z)$ and $\tilde{\mathcal{V}}_f(\bar{z})$ are conformal primaries of the matter theory with $L_0 = \tilde{L}_0 = 0$ and with $\tilde{\mathcal{V}}_f$ in the Ramond sector. One then needs to write the massless particle vertex in the -1 picture,

$$\mathcal{V}(z, \bar{z}) = \sqrt{\frac{2}{\alpha'}} c \tilde{c} e^{-\tilde{\phi}} i \partial X_L^l \tilde{\psi}^j e^{i\mathbf{q} \cdot (\mathbf{X}_L + \mathbf{X}_R)} .$$

The holomorphic part of the correlator is identical, giving $C = \sqrt{\varepsilon(r_\varepsilon^l + 1)}$. The antiholomorphic part reads

$$\tilde{C} = \bar{z}_{12} \bar{z}_{23} \bar{z}_{13} \left\langle e^{-\tilde{\phi}/2} \tilde{\mathcal{V}}_f^*(\bar{z}_3) e^{-\tilde{\phi}} \tilde{\psi}^j e^{i\mathbf{q} \cdot \mathbf{X}_R(\bar{z}_2)} e^{-\tilde{\phi}/2} u^\alpha \tilde{\Theta}_\alpha e^{i\mathbf{p}_R \cdot \mathbf{X}_R(\bar{z}_1)} \right\rangle ,$$

with the relevant DDF operator now given by

$$\tilde{A}_{-\varepsilon}^j \equiv \oint \frac{d\bar{z}}{2\pi i} e^{-\tilde{\phi}} \tilde{\psi}^j e^{i\mathbf{q} \cdot \mathbf{X}_R(\bar{z})}$$

for $\mathbf{q} = -\varepsilon \mathbf{k}$ and integral ε . As before, \tilde{A}_ε^j annihilates the initial state and $[\tilde{A}_\varepsilon^j, \tilde{A}_{-\varepsilon}^j] = \varepsilon$ determines the constant $\tilde{C} = \sqrt{\varepsilon}$.

3.3 Absorption cross section

The absorption cross section is given by

$$\sigma_{abs} = \frac{\pi}{4M\omega(M+\omega)} \sum_{M'} \delta\left(\sqrt{M'^2 + \omega^2} - M - \omega\right) |\mathcal{M}|^2 ,$$

where ω is the massless particle energy in the massive string reference frame and the amplitude \mathcal{M} is given by (3.5). Using the kinematical relation (3.4) one can simplify the absorption cross section expression to

$$\sigma_{abs} = \frac{\pi\alpha'}{8M\omega} \sum_\varepsilon \delta\left(\frac{\alpha'}{2}M\omega - \varepsilon\right) |\mathcal{M}|^2 ,$$

so that, for a given microstate, one has

$$\sigma_{abs} = \pi\kappa^2 \sum_\varepsilon \varepsilon \left(r_\varepsilon^l + 1\right) \delta\left(\frac{\alpha'}{2}M\omega - \varepsilon\right) .$$

The black hole should be thought of as a classical mixture of all its microstates. In appendix B we show that

$$\langle r_\varepsilon^l \rangle = \frac{1}{\exp\left(\frac{2\pi\varepsilon}{\sqrt{N}}\right) - 1} ,$$

for $N \gg 1$ and where the average $\langle \rangle$ is taken over all left-moving microstates of the BPS string with charges n and m . Thus, recalling that $2\kappa^2 = 2\pi g^2 \alpha'$, we have

$$\langle \sigma_{abs} \rangle = \pi^2 g^2 \alpha' \sum_{\varepsilon} \frac{\varepsilon}{1 - \exp\left(-\frac{2\pi\varepsilon}{\sqrt{N}}\right)} \delta\left(\frac{\alpha'}{2} M\omega - \varepsilon\right).$$

For energies much larger than the black hole mass gap,

$$\omega \gg \frac{2}{\alpha' M},$$

we can drop the sum of delta functions. Notice that this condition always holds in the classical limit, since $\alpha' M \rightarrow \infty$ and therefore the mass gap vanishes.⁵ The last expression can then be recasted in the form

$$\langle \sigma_{abs} \rangle = \frac{\pi}{2} g^2 \alpha' \sqrt{N} \frac{\omega}{2T_L} \frac{1}{1 - \exp\left(-\frac{\omega}{2T_L}\right)},$$

where

$$T_L = \frac{\sqrt{N}}{2\pi\alpha' M}$$

is the effective left-moving temperature. This result is universal in the sense that it does *not* depend on the massless particle polarization.

To compare with the gravity phase we must consider the low energy limit, $\omega \ll T_L$, where

$$\langle \sigma_{abs} \rangle \simeq \frac{\pi}{2} g^2 \alpha' \sqrt{N} \simeq G_N S.$$

This result clearly indicates the difference between string and black hole phases. Indeed, we saw in the gravity phase that the minimally coupled scalars and the fixed scalars absorb quite differently. In particular, in the limit $\omega \rightarrow 0$, the fixed scalars do not absorb since their values are fixed at the horizon. A finite cross section for *all* particle polarizations signals therefore the absence of a horizon for small values of the coupling. Figure 2 of the Introduction schematically summarizes this discussion.

3.4 String size

In this last section, we estimate the size of the BPS massive string state as a function of the string coupling, following [3].⁶ Having chosen the center-of-mass frame for the initial state with $p^\mu = 0$ for $\mu = 1, 2, 3$, we can define the size \mathcal{R} of a given microstate $|\mathcal{V}_i\rangle$ via

$$\mathcal{R}^2 \equiv \langle \mathcal{V}_i | (X^\mu - X_{\text{CM}}^\mu)^2 | \mathcal{V}_i \rangle = \alpha' \sum_{m=1}^{\infty} \frac{1}{m^2} \langle \mathcal{V}_i | \alpha_{-m}^\mu \alpha_m^\mu + \tilde{\alpha}_{-m}^\mu \tilde{\alpha}_m^\mu | \mathcal{V}_i \rangle + \alpha' \sum_{m=1}^{\infty} \frac{1}{m},$$

⁵There are other excitations of the BPS string, particularly those associated to the addition of $NS5/\overline{NS5}$ brane pairs [24]. Although for $g^2 \sqrt{N} \gg 1$ the mass gap for these excitations is smaller than the fundamental string excitations mass gap, $2/(\alpha' M)$, these are entropically disfavoured provided the energy ω is kept finite.

⁶The main differences are that here we consider BPS states and that, by parameterizing the states with DDF operators, we have full control of the Virasoro constraints.

where X_{CM}^μ are the zero modes of the string and where

$$\alpha_m^\mu = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X_L^\mu(z)$$

are the usual raising and lowering operators. The physical meaning of the divergent sum, which comes from normal ordering the operators, is discussed in [33]. This is a state independent effect that we shall discard. The BPS states under consideration do not have right-moving excitations so that the $\tilde{\alpha}$ contribution to \mathcal{R}^2 is zero. Then, we have

$$\mathcal{R}^2 = \alpha' \sum_{m=1}^{\infty} \frac{1}{m^2} \Delta_m, \tag{3.6}$$

where $\Delta_m \equiv \langle \mathcal{V}_i | \alpha_{-m}^\mu \alpha_m^\mu | \mathcal{V}_i \rangle$.

The parametrization of the microstates \mathcal{V}_i using DDF operators explained in section 3.1 breaks rotational invariance because one has to select a spatial direction for the null vector \mathbf{k} . Nevertheless, after averaging over all possible microstates the rotational symmetry should be recovered. Therefore, for simplicity, we compute the average size of the string in a direction X^μ ($\mu = 2, 3$) orthogonal to \mathbf{k} . In appendix C we show explicitly that the same result holds if we choose the spatial direction of the null vector \mathbf{k} . For each left-moving microstate

$$|\mathcal{V}_i\rangle = \prod_{i=2}^{25} \prod_{n=1}^{\infty} \frac{1}{\sqrt{r_n^i! n^{r_n^i}}} (A_{-n}^i)^{r_n^i} |\mathbf{p}_L + N\mathbf{k}\rangle,$$

we compute Δ_m by using the commutation relations

$$[\alpha_m^\mu, A_n^i] = m\delta^{\mu i} B_m^n, \quad [A_n^i, B_m^l] = [B_m^n, B_p^l] = 0,$$

where

$$B_m^n = \oint \frac{dz}{2\pi i} z^{m-1} e^{i n \mathbf{k} \cdot \mathbf{X}_L(z)}.$$

We obtain the expression

$$\Delta_m = \left| \sum_{l=1}^{\infty} \frac{m r_l^\mu}{\sqrt{r_l^\mu! l^{r_l^\mu}}} (A_{-l}^\mu)^{r_l^\mu-1} \prod_{n \neq l} \frac{1}{\sqrt{r_n^\mu! n^{r_n^\mu}}} (A_{-n}^\mu)^{r_n^\mu} B_m^{-l} |\mathbf{p}_L + N\mathbf{k}\rangle \right|^2.$$

Since $B_m^{-l} |\mathbf{p}_L + N\mathbf{k}\rangle = 0$ for $l < m$ and $(B_m^{-l})^\dagger = B_{-m}^l$ the sum over l reduces to the single term $l = m$. Finally, using $B_m^{-m} |\mathbf{p}_L + N\mathbf{k}\rangle = |\mathbf{p}_L + (N - m)\mathbf{k}\rangle$, we obtain the intuitive result

$$\Delta_m = m r_m^\mu.$$

Averaging over the microstates and in the large N limit (see appendix B) we have

$$\langle \Delta_m \rangle = \frac{m}{\exp \frac{2\pi m}{\sqrt{N}} - 1}.$$

The sum in (3.6) is dominated by small values of m . Thus we only need

$$\langle \Delta_m \rangle \simeq \frac{\sqrt{N}}{2\pi},$$

for $m \ll \sqrt{N}$, obtaining

$$\langle \mathcal{R}^2 \rangle \simeq \frac{\pi}{12} \alpha' \sqrt{N},$$

for large N .

Besides the average string size one might want to know how many string states have a given size. This size distribution is described by an entropy function $S_0(N, \mathcal{R})$. Following [3] we show in appendix B that, in the large N limit,

$$S_0(N, \mathcal{R}) \simeq 4\pi\sqrt{N} \left(1 - \frac{\alpha'}{32\mathcal{R}^2} \right), \quad \alpha' \ll \mathcal{R}^2 \ll \alpha'\sqrt{N},$$

and

$$S_0(N, \mathcal{R}) \simeq 4\pi\sqrt{N} \left(1 - \frac{\mathcal{R}^2}{2\alpha'N} \right), \quad \alpha'\sqrt{N} \ll \mathcal{R}^2 \ll \alpha'N.$$

These results suggest a description of the free string as a random walk with \sqrt{N} steps and step size of order $\sqrt{\alpha'}$.

3.4.1 Self-gravitating string

We would like to estimate the variation of the string size when the string coupling is increased. We have just seen that at zero coupling the string is, on average, very large ($\mathcal{R}^2 \sim \alpha'\sqrt{N}$) and can be described as a random polymer. On the other hand, for large enough g , we expect the string to collapse into the black hole described in section 2, whose size is of order $\sqrt{\alpha'}$. Therefore, as we vary the coupling, we expect the string to contract due to gravitational interactions.

To study the contracting string for small g we shall use the polymer picture of a low interacting string [34]. This model was used to describe non-BPS string states, which, as the string contracts, decrease their mass. Although for BPS states the mass will remain constant, the same physical effect will reduce the string size. The idea is to deform the ensemble at $g = 0$ by adding to the entropy $S_0(N, \mathcal{R})$ a term describing the gravitational interaction between the \sqrt{N} bits of the polymer, so that

$$S_g(N, \mathcal{R}) \sim 4\pi\sqrt{N} - \frac{\pi}{8}\sqrt{N} \frac{\alpha'}{\mathcal{R}^2} + c g^2 N \frac{\sqrt{\alpha'}}{\mathcal{R}},$$

for large N and $\alpha' \ll \mathcal{R}^2 \ll \alpha'\sqrt{N}$. The four-dimensional gravitational character of the interaction fixes the \mathcal{R} dependence and the g^2 factor in the last term. Finally, c is a positive number of order one. The value of \mathcal{R} that maximizes S_g is given by

$$\langle \mathcal{R}^2 \rangle \sim \frac{\alpha'}{g^4 N}. \tag{3.7}$$

This should be a rough estimate of the string size for $N^{-3/4} \ll g^2 \ll N^{-1/2}$.

We conclude that, as we vary the coupling constant g , the string has three different phases. For $g^2 \ll N^{-3/4}$ the string is well described by a free polymer and $\langle \mathcal{R}^2 \rangle \sim \alpha' \sqrt{N}$. In the intermediate region $N^{-3/4} \ll g^2 \ll N^{-1/2}$, the string is well described by a self-interacting polymer and its average size is given by (horizon radius squared $\mathcal{R}_h^2 = \alpha'/2$, as explained in section 2. The three phases are schematically plotted as a function of the coupling in figure 1 of the Introduction.

4. Conclusion

In this work we studied the string/black hole transition in the controlled setting of BPS string states of the Heterotic string. We have shown evidence that, by increasing the string coupling, free winding string states evolve into small black holes. There are several computations that would confirm the physical picture here discussed.

The gravity description, including all α' corrections, is exact in the classical limit $g \rightarrow 0$ with the classical charges ρ_n and ρ_m fixed. In this limit the coupling $\lambda = g^2 \sqrt{N}$ is finite and $N \rightarrow \infty$. Therefore the free string regime corresponds to $\lambda \equiv 0$ and infinite string size, as can be seen from figure 1. On the other hand, the self-gravitating string phase, for which $\lambda \ll 1$, should be seen in the gravity framework. This fact allows for a direct test of the string/black hole transition by studying the interpolating solution between the asymptotic flat spacetime (2.1) and the $AdS_2 \times S^2$ near-horizon geometry (2.3). For large enough λ the black hole solution exists and has a *charge independent* horizon radius in the string frame. Below some critical value $\lambda_c \sim 1$ we enter the self-gravitating string domain and no longer expect the interpolating solution to exist. Note that, when $\lambda \sim \lambda_c$, the asymptotic value of the coupling at infinity is of the order of the fixed value at the horizon. Moreover the string coupling will always start to decrease as we move towards the black hole. Therefore, if an interpolating solution exists for $\lambda \ll \lambda_c$ its string coupling must be a non-monotonic function of the radial variable. These seems unlikely to us even though we do not have a compelling argument. It would be very nice to check this conjecture.

Another check of the transition is to determine the behaviour of the order parameter $\sigma_{abs}(\omega = 0)$ away from the free string phase. This would require the computation of the 1-loop absorption amplitude. It would be nice to see the splitting of the universal result $G_N S$ for different polarizations, as schematically presented in figure 2.

There are excitations of the BPS string associated to the creation of $NS5/\overline{NS5}$ brane pairs that are important when considering the near extremal version of the system [24]. In fact, the mass gap for these excitations decreases as the gravitational coupling is increased. Close to (or possibly at) the string/black hole transition the $NS5/\overline{NS5}$ pairs mass gap crosses the fundamental string excitations mass gap. This could mean that in our computations of the cross section the creation of $NS5/\overline{NS5}$ pairs should have been considered. However, this is not the case. For finite energies of the incoming particle, the fundamental string excitations are entropically favoured so that they dominate the dynamics of the absorption process. On the other hand, when considering the *classical* near extremal version of the black hole the $NS5/\overline{NS5}$ excitations are favoured. In this limit the energy ΔE above the BPS ground state is infinite, with a fixed ratio with respect to the black hole

mass. It would be very interesting to study the near extremal version of this black hole and corresponding transition.

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A. Barrier penetration at low energies

In this appendix we will study the problem of barrier penetration for very small energy. We will present a general method to compute the small energy behaviour of the transmission amplitude through a one–dimensional potential V which tends to zero at infinity.

The Schrödinger equation for a wave function Ψ is

$$\left(\omega^2 + \frac{d^2}{dx^2}\right) \Psi(x) = V(x) \Psi(x) .$$

The general idea will be to solve this equation using different approximations in different regions and then glue the various solutions in the overlapping zones. Firstly, let us define region I (see figure 4) by the region where one can neglect the ω^2 term in Schrödinger equation. This can be done as long as $\omega^2 \ll V(x)$ or equivalently for $x_- \ll x \ll x_+$, where x_{\pm} are the classical turning points defined by $V(x_{\pm}) = \omega^2$. Notice that the size of this region diverges as $\omega \rightarrow 0$ because the potential goes to zero at infinity. Secondly, if l is the largest length scale of the potential V , then for $|x| \gg l$ the potential is well described by its dominant asymptotic behaviour. We shall call these regions II and III, respectively for negative and positive x . For small enough ω the classical turning points, x_{\pm} , are deep inside regions II and III and one has large overlapping zones B and C with region I. We also define regions A and D as the regions where $V(x) \ll \omega^2$ (see figure 4). Let us remark that, the following analysis is general enough to allow for *any* linear equation of motion in the middle of region I, not necessarily of the Schrödinger type.

For each region A, B, C and D there are two natural independent solutions of the Schrödinger equation. In the asymptotic regions A and D the natural choice are the WKB wave functions

$$\begin{aligned} \Psi_1^{A,D}(x) &\simeq \sqrt{\frac{\omega}{p(x)}} \exp\left(i \int_{x_{\mp}}^x p(x') dx'\right) , \\ \Psi_2^{A,D}(x) &\simeq \sqrt{\frac{\omega}{p(x)}} \exp\left(-i \int_{x_{\mp}}^x p(x') dx'\right) , \end{aligned}$$

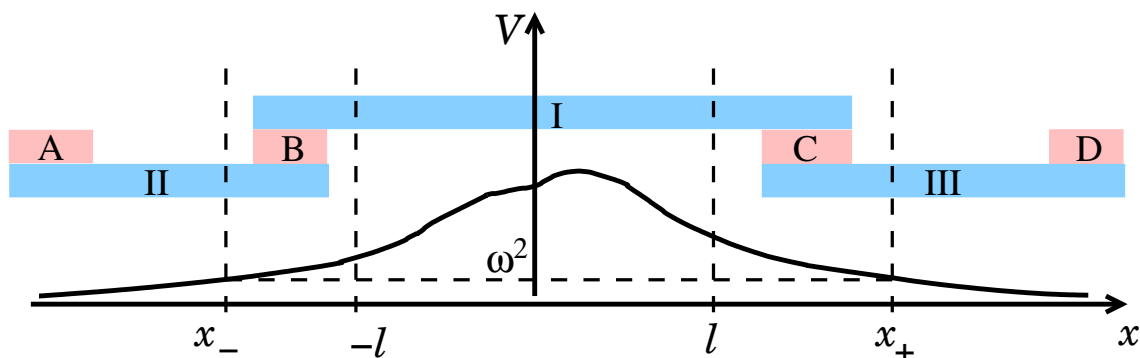


Figure 4: Schrödinger potential. The classical turning points are x_+ and x_- and l is the largest length scale of the potential. Region I is defined by $\omega^2 \ll V(x)$ so that one can set $\omega = 0$ in the Schrödinger equation. Regions II and III are the regions where the potential is well approximated by its dominant asymptotic behaviour. Regions B and C are the intersection of region I and regions II and III, respectively. In regions A and D the potential is much smaller than ω^2 .

where $p(x) \equiv \sqrt{\omega^2 - V(x)}$. The scattering problem is to find the matrix M such that

$$\Psi_i^A = M_{ij} \Psi_j^D .$$

Then

$$M = \begin{pmatrix} \frac{1}{T} & \frac{R}{T} \\ \frac{R^*}{T^*} & \frac{1}{T^*} \end{pmatrix} ,$$

where T and R are the transmission and reflection coefficients which satisfy $|T|^2 + |R|^2 = 1$. Our method consists in writing M as the product $M^{II} M^I M^{III}$ where

$$\Psi_i^A = M_{ij}^{II} \Psi_j^B , \quad \Psi_i^B = M_{ij}^I \Psi_j^C , \quad \Psi_i^C = M_{ij}^{III} \Psi_j^D .$$

The natural independent solutions $\Psi^{B,C}$ in regions B and C depend on the asymptotic form of the potential but not on the energy ω^2 because in these regions the energy is negligible compared to the potential. Therefore the matrix M^I is also independent of ω since it is obtained from solving Schrödinger equation for $\omega = 0$ and gluing Ψ^B with Ψ^C . On the other hand, M^{II} and M^{III} depend on ω and only on the asymptotic behaviour of the potential. Thus, we see that the low energy behaviour of the transmission coefficient is essentially controlled by the asymptotic form of the potential. We now study the Schrödinger equation in regions II and III for different asymptotic behaviours of the potential V .

We consider first the case where $V(x)$ decays faster than $1/x^2$ at infinity. In this case the asymptotic wave functions Ψ^A and Ψ^D are just the usual plane waves

$$\Psi_1^{A,D}(x) \simeq e^{i\omega x} , \quad \Psi_2^{A,D}(x) \simeq e^{-i\omega x} ,$$

up to some phase shift that goes to zero as ω tends to zero. Studying the asymptotics of the Schrödinger equation with $\omega = 0$ it is easy to see that the natural independent solutions in regions B and C have the following leading behaviour

$$\Psi_1^{B,C}(x) \simeq 1 , \quad \Psi_2^{B,C}(x) \simeq x .$$

For these potentials, the distance between 0 and the classical turning points x_{\pm} grows slower than the asymptotic wave length $2\pi/\omega$ as ω tends to zero. The size of the transition interval $[a, b]$, where the potential influences the wave function between regions A and B (or regions C and D), is of the order of x_- (or x_+). This means that the wave function essentially keeps its asymptotic plane wave form until region B (or C). Another way to see this is to estimate the variation in the derivative of the wave function through the transition interval

$$\left. \frac{d\Psi}{dx} \right|_b - \left. \frac{d\Psi}{dx} \right|_a = \int_a^b (V(x) - \omega^2) \Psi(x) dx \sim \omega^2 x_- \ll \omega \sim \left. \frac{d\Psi}{dx} \right|_a.$$

Furthermore, regions B and C from the point of view of the plane waves are the same as $x \simeq 0$, so that the matching can be done using $\exp(\mp i\omega x) \simeq 1 \mp i\omega x$, yielding

$$M^{II} = \begin{pmatrix} 1 & i\omega \\ 1 & -i\omega \end{pmatrix},$$

and

$$M^{III} = (M^{II})^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i/\omega & i/\omega \end{pmatrix}.$$

If the potential decays slower than $1/x^2$ the WKB approximation is applicable [35]. Away from the turning points the WKB approximation requires

$$|V'(x)| \ll |p(x)|^3,$$

where prime stands for $\frac{d}{dx}$. This condition is always satisfied in regions A and D, however, only if the potential decays slower than $1/x^2$ it is also verified in regions B and C. In this case, we choose

$$\begin{aligned} \Psi_1^{B,C}(x) &\simeq V^{-1/4}(x) \exp\left(\pm \int_0^x dx' \sqrt{V(x')}\right), \\ \Psi_2^{B,C}(x) &\simeq V^{-1/4}(x) \exp\left(\mp \int_0^x dx' \sqrt{V(x')}\right). \end{aligned}$$

To glue the solutions across the turning points x_{\pm} we use the standard linear approximation of the potential $V(x) \simeq \omega^2 + V'(x_{\pm})(x - x_{\pm})$ and solve the Schrödinger equation for this linear potential. To be able to use the linearized potential to glue the WKB wave functions on the two sides of the turning point, we need

$$|V''(x_{\pm})| \ll |V'(x_{\pm})|^{4/3}.$$

In the small ω limit this is equivalent to the fact that the potential decays slower than $1/x^2$ as x goes to infinity.⁷ The general solution around the turning point x_- is given by a linear combination of the Airy functions $Ai(z)$ and $Bi(z)$, where $z = |V'(x_-)|^{1/3}(x - x_-)$.

⁷Here we are assuming that the potential is not pathological like for example $x^{-1} \sin^2(x^2)$.

Matching the asymptotics of the Airy functions with the WKB wave functions close to the turning point we obtain

$$\Psi_{1,2}^A(x) \simeq \sqrt{\pi\omega} |V'(x_-)|^{-1/6} e^{\pm i\pi/4} (Bi(z) \mp iAi(z)) , \quad x \simeq x_- ,$$

and

$$\begin{aligned} \Psi_1^B(x) &\simeq e^{-P_-} |V'(x_-)|^{-1/6} \sqrt{\pi} Bi(z) , \\ \Psi_2^B(x) &\simeq e^{P_-} |V'(x_-)|^{-1/6} 2\sqrt{\pi} Ai(z) , \quad x \simeq x_- , \end{aligned}$$

where $P_- = \int_{x_-}^0 dx' |p(x')|$. Thus,

$$M^{II} = e^{-i\pi/4} \frac{\sqrt{\omega}}{2} \begin{pmatrix} 2ie^{P_-} & e^{-P_-} \\ 2e^{P_-} & ie^{-P_-} \end{pmatrix} .$$

Similarly

$$M^{III} = \frac{e^{-i\pi/4}}{2\sqrt{\omega}} \begin{pmatrix} ie^{-P_+} & e^{-P_+} \\ 2e^{P_+} & 2ie^{P_+} \end{pmatrix} ,$$

with $P_+ = \int_0^{x_+} dx' |p(x')|$.

Finally, we consider the marginal situation where the potential has the asymptotic form

$$V(x) \sim \frac{U_{\pm}}{x^2} , \quad x \rightarrow \pm\infty .$$

In this case, Ψ^A and Ψ^D tend to plane waves with an extra phase

$$\Psi_1^{A,D}(x) \simeq \exp\left(i\omega x \pm i\frac{\pi}{2}\sqrt{U_{\mp}}\right) , \quad \Psi_2^{A,D}(x) \simeq \exp\left(-i\omega x \mp i\frac{\pi}{2}\sqrt{U_{\mp}}\right) .$$

The natural choices for Ψ^B and Ψ^C are

$$\Psi_1^{B,C}(x) \simeq x^{1/2-\beta_{\mp}} , \quad \Psi_2^{B,C}(x) \simeq x^{1/2+\beta_{\mp}} ,$$

where $U_{\pm} = \beta_{\pm}^2 - 1/4$. For this specific asymptotic behaviour of the potential the solutions in regions II and III can be written in terms of Bessel functions. Using the asymptotic expansions of the Bessel functions, one determines the wave functions in region II

$$\begin{aligned} \Psi_{1,2}^A(x) &\simeq -\frac{e^{\pm i\phi_{\mp}^-}}{\sin(\beta_{\mp}-\pi)} \sqrt{\frac{\pi\omega x}{2}} \left[J_{\beta_{\mp}}(\omega x) - e^{\pm i\beta_{\mp}-\pi} J_{-\beta_{\mp}}(\omega x) \right] , \\ \Psi_{1,2}^B(x) &\simeq \Gamma(1 \mp \beta_{\mp}) \left(\frac{2}{\omega}\right)^{1/2 \mp \beta_{\mp}} \sqrt{\frac{\omega x}{2}} J_{\mp\beta_{\mp}}(\omega x) , \end{aligned}$$

where we define $\phi_a^{\pm} = \frac{\pi}{2}(\sqrt{U_a} \pm \beta_a - 1/2)$ with the index $a = \pm$. This yields

$$M^{II} = \begin{pmatrix} e^{i\phi_{\mp}^+} f(\beta_{\mp}) & e^{i\phi_{\mp}^-} f(-\beta_{\mp}) \\ e^{-i\phi_{\mp}^+} f(\beta_{\mp}) & e^{-i\phi_{\mp}^-} f(-\beta_{\mp}) \end{pmatrix} ,$$

with $f(\beta) = \frac{1}{\sqrt{\pi}}\Gamma(\beta) \left(\frac{\omega}{2}\right)^{\frac{1}{2}-\beta}$. A similar calculation gives

$$M^{III} = \frac{1}{2} \begin{pmatrix} e^{i\phi_+^+} f(1 - \beta_+) & e^{-i\phi_+^+} f(1 - \beta_+) \\ e^{i\phi_+^-} f(1 + \beta_+) & e^{-i\phi_+^-} f(1 + \beta_+) \end{pmatrix}.$$

In all cases, the dominant low energy behaviour of the transmission probability is given by

$$|T|^2 \simeq |M_{11}^{II}|^{-2} |M_{21}^{III}|^{-2} |M_{12}^I|^{-2}, \tag{A.1}$$

from which we conclude that

$$|T|^2 \sim \omega^{2\beta_- + 2\beta_+},$$

if the potential decays as $1/x^2$. If it decays faster than $1/x^2$ for $x \rightarrow \pm\infty$, one should replace $\omega^{2\beta_{\pm}}$ by ω . If it decays slower than $1/x^2$, the factor $\omega^{2\beta_{\pm}}$ should be replaced by $\exp(\mp 2 \int_0^{x_{\pm}} dx' |p(x')|)$. If it decays slower on both sides one recovers the usual WKB tunneling factor

$$|T|^2 \sim \exp\left(-2 \int_{x_-}^{x_+} dx' \sqrt{V(x') - \omega^2}\right).$$

As we have just seen, the ω dependence of the low energy transmission probability is independent of the details of the potential, uniquely relying on its asymptotics. However, if one is interested in the precise numerical factors, it is enough to determine $|M_{12}^I|^2$ for some given potential and use formula (A.1). It is then clear that this method allows *any* linear evolution in the middle region since we only need the ω independent numerical coefficient M_{12}^I .

B. Averages over string states

In the main text we needed the average occupation number $\langle r_n^l \rangle$, in the ensemble of all partitions of the integer N in integers of 24 colours. More precisely, an element of this ensemble is the set of non-negative integers $\{r_m^i\}$ obeying the constraint

$$\sum_{i=1}^{24} \sum_{m=1}^{\infty} m r_m^i = N. \tag{B.1}$$

We are interested in the average $\langle r_n^l \rangle_N$ for fixed and large N . In this limit, one can trade the constraint (B.1) by a chemical potential μ

$$\langle r_n^l \rangle_{\mu} = \frac{1}{Z} \sum_{\{r_m^i\}} r_n^l e^{-\mu \sum_{i=1}^{24} \sum_{m=1}^{\infty} m r_m^i} = \frac{1}{e^{\mu n} - 1},$$

where $Z = \sum_{\{r_m^i\}} \exp\left(-\mu \sum_{i=1}^{24} \sum_{m=1}^{\infty} m r_m^i\right)$ is the usual partition function. In the limit of large N (or small μ) the averages $\langle \rangle_{\mu}$ and $\langle \rangle_N$ are related through

$$\langle N \rangle_{\mu} = 24 \sum_{n=1}^{\infty} \frac{n}{e^{\mu n} - 1} \simeq \frac{24}{\mu^2} \int_0^{\infty} \frac{x dx}{e^x - 1} = \frac{4\pi^2}{\mu^2}.$$

Thus

$$\langle r_n^l \rangle_N = \frac{1}{\exp\left(\frac{2\pi n}{\sqrt{N}}\right) - 1}.$$

Following [3] we will estimate the number of string states with a given size

$$\mathcal{R}^2 = \alpha' \sum_{m=1}^{\infty} \frac{1}{m} r_m^i.$$

To this end we introduce a chemical potential γ conjugated to the size of the string. Defining

$$e^{F(\mu, \gamma)} \equiv \sum_{\{r_m^i\}} e^{-\mu N - \gamma \mathcal{R}^2 / \alpha'},$$

we obtain the entropy as a function of \mathcal{R} from the Legendre transform

$$S(N, \mathcal{R}) = F(\mu, \gamma) + \mu N + \gamma \mathcal{R}^2 / \alpha',$$

where N and \mathcal{R} are related to μ and γ through

$$N = -\frac{\partial F}{\partial \mu}, \quad \mathcal{R}^2 = -\alpha' \frac{\partial F}{\partial \gamma}.$$

The free energy F is then given by

$$F(\mu, \gamma) = -23 \sum_{n=1}^{\infty} \log(1 - e^{-\mu n}) - \sum_{n=1}^{\infty} \log(1 - e^{-\mu n - \gamma/n}). \quad (\text{B.2})$$

Let us first consider the region of parameters $\mu \ll \sqrt{\mu\gamma} \ll 1$. In this case the sums in F can be well approximated by integrals

$$F(\mu, \gamma) \simeq -\frac{23}{\mu} \int_0^{\infty} dx \log(1 - e^{-x}) - \frac{1}{\mu} \int_0^{\infty} dx \log(1 - e^{-x - \mu\gamma/x}),$$

so that

$$F(\mu, \gamma) \simeq \frac{4\pi^2}{\mu} - \pi \sqrt{\frac{\gamma}{\mu}} + O(\gamma).$$

The entropy is then

$$S(N, \mathcal{R}) = 4\pi\sqrt{N} \left(1 - \frac{\alpha'}{32\mathcal{R}^2}\right),$$

for large N and $\alpha' \ll \mathcal{R}^2 \ll \alpha'\sqrt{N}$.

To determine the entropy function for values of \mathcal{R}^2 larger than the typical value $\alpha'\sqrt{N}$ we need to consider negative chemical potential γ . The free energy (B.2) is well defined as long as $\gamma > -\mu$. For $0 < \gamma + \mu \ll \mu \ll 1$,

$$F(\mu, \gamma) \simeq \frac{4\pi^2}{\mu} - \log(\gamma + \mu) + O(1),$$

where the γ dependence steams from the first term in the sum. Therefore

$$S(N, \mathcal{R}) = 4\pi\sqrt{N} \left(1 - \frac{\mathcal{R}^2}{2\alpha'N}\right),$$

for large N and $\alpha'\sqrt{N} \ll \mathcal{R}^2 \ll \alpha'N$.

C. Average string size

In this appendix we compute the average size of the string in the spatial direction X^μ ($\mu = 1$) of the null vector \mathbf{k} appearing in the DDF operators. In this case we have different commutation relations

$$[\alpha_m^\mu, A_n^i] = n\sqrt{\frac{\alpha'}{2}} k^\mu D_{m,n}^i, \quad [D_{m,n}^i, A_l^j] = -l\delta^{ij} E_m^{n+l},$$

where

$$D_{m,n}^i = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^i(z) e^{in\mathbf{k}\cdot\mathbf{X}_L(z)},$$

and

$$E_m^n = \oint \frac{dz}{2\pi} z^m \mathbf{k} \cdot \partial \mathbf{X}_L(z) e^{in\mathbf{k}\cdot\mathbf{X}_L(z)}.$$

Furthermore

$$[D_{-m,n}^i, D_{m,-l}^j] = \delta^{ij} (n\delta_{n,l} - mB_0^{n-l}), \quad [E_m^n, A_l^i] = [E_m^n, D_{m,l}^i] = 0.$$

Commuting the α_m^μ we obtain

$$\Delta_m = \frac{\alpha'}{2} (k^\mu)^2 \left| \sum_i \sum_{n=1}^{\infty} \sum_{s=1}^{r_n^i} \prod_{l=1}^{n-1} \frac{(A_{-l}^i)^{r_l^i} (A_{-n}^i)^{r_n^i - s} n D_{m,-n}^i (A_{-n}^i)^s}{\sqrt{r_l^i! l^{r_l^i}} \sqrt{r_n^i! n^{r_n^i}}} \prod_{l=n+1}^{\infty} \frac{(A_{-l}^i)^{r_l^i}}{\sqrt{r_l^i! l^{r_l^i}}} |\mathbf{p}_L + N\mathbf{k}\rangle \right|^2.$$

Since $D_{m,-n}^i |\mathbf{p}_L + N\mathbf{k}\rangle = 0$ for $n < m$ we commute the $D_{m,-n}^i$ with $n < m$ to the right and the ones with $n > m$ to the left. In this process we will generate many E_m^{-n-l} operators. However, because these commute with all the other operators and satisfy $E_m^{-n} |\mathbf{p}_L + N\mathbf{k}\rangle = 0$ for $n < m$ and $(E_m^{-n})^\dagger = E_{-m}^n$, only the E_m^{-m} operators remain. The contribution of the latter is easily computed using $E_m^{-m} |\mathbf{p}_L + N\mathbf{k}\rangle = |\mathbf{p}_L + (N-m)\mathbf{k}\rangle$. Finally, using $D_{m,-m}^i |\mathbf{p}_L + N\mathbf{k}\rangle = \sqrt{\alpha'/2} p_L^i |\mathbf{p}_L + (N-m)\mathbf{k}\rangle$, we arrive at

$$\begin{aligned} \Delta_m &= \frac{\alpha'}{2} (k^\mu)^2 \sum_i \left(\frac{\alpha'}{2} (p_L^i)^2 m r_m^i + \sum_{n+l=m} n l r_n^i r_l^i + \sum_{n=m+1}^{\infty} n r_n^i (n-m) + \sum_{n-l=m} n l r_n^i r_l^i \right) \\ &= m r_m^4 + \frac{2}{\alpha' M^2} \left[\sum_i \left(\sum_{n+l=m} n l r_n^i r_l^i + \sum_{n=m+1}^{\infty} n r_n^i (n-m) + \sum_{n-l=m} n l r_n^i r_l^i \right) - 2(N-1) m r_m^4 \right]. \end{aligned}$$

After averaging Δ_m over the microstates one should obtain the same result as for the other directions,

$$\langle \Delta_m \rangle = m \langle r_m^i \rangle,$$

for any N and m . This implies the identity

$$\sum_{n=1}^m n(m-n) \langle r_n^i r_{m-n}^i \rangle + \sum_{n=m+1}^{\infty} n(n-m) \langle r_n^i (r_{n-m}^i + 1) \rangle = \frac{N-1}{12} m \langle r_m^i \rangle. \quad (\text{C.1})$$

In the limit $m \ll \sqrt{N}$, it is easy to see that both sides of (C.1) have the same dominant behaviour $N^{3/2}/(24\pi)$. This ensures that

$$\langle \Delta_m \rangle \simeq \frac{\sqrt{N}}{2\pi},$$

as expected.

References

- [1] L. Susskind, *Some speculations about black hole entropy in string theory*, hep-th/9309145.
- [2] G.T. Horowitz and J. Polchinski, *A correspondence principle for black holes and strings*, *Phys. Rev. D* **55** (1997) 6189 [hep-th/9612146];
G.T. Horowitz and J. Polchinski, *Self gravitating fundamental strings*, *Phys. Rev. D* **57** (1998) 2557 [hep-th/9707170].
- [3] T. Damour and G. Veneziano, *Self-gravitating fundamental strings and black holes*, *Nucl. Phys. B* **568** (2000) 93 [hep-th/9907030].
- [4] A. Dabholkar, G.W. Gibbons, J.A. Harvey and F. Ruiz Ruiz, *Superstrings and solitons*, *Nucl. Phys. B* **340** (1990) 33.
- [5] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett. B* **451** (1999) 309 [hep-th/9812082].
- [6] R.M. Wald, *Black hole entropy in the noether charge*, *Phys. Rev. D* **48** (1993) 3427 [gr-qc/9307038].
- [7] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301 [hep-th/0409148].
- [8] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett. A* **10** (1995) 2081 [hep-th/9504147].
- [9] R.R. Khuri and R.C. Myers, *Dynamics of extreme black holes and massive string states*, *Phys. Rev. D* **52** (1995) 6988 [hep-th/9508045].
- [10] C.G. Callan Jr., J.M. Maldacena and A.W. Peet, *Extremal black holes as fundamental strings*, *Nucl. Phys. B* **475** (1996) 645 [hep-th/9510134].
- [11] G. Mandal and S.R. Wadia, *Black hole geometry around an elementary BPS string state*, *Phys. Lett. B* **372** (1996) 34 [hep-th/9511218].
- [12] A. Sen, *How does a fundamental string stretch its horizon?*, *JHEP* **05** (2005) 059 [hep-th/0411255].
- [13] V. Hubeny, A. Maloney and M. Rangamani, *String-corrected black holes*, *JHEP* **05** (2005) 035 [hep-th/0411272].
- [14] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287].
- [15] D. Bak, S. Kim and S.-J. Rey, *Exactly soluble BPS black holes in higher curvature $N = 2$ supergravity*, hep-th/0501014.
- [16] A. Sen, *Black holes, elementary strings and holomorphic anomaly*, *JHEP* **07** (2005) 063 [hep-th/0502126].
- [17] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Exact and asymptotic degeneracies of small black holes*, *JHEP* **08** (2005) 021 [hep-th/0502157].
- [18] A. Sen, *Black holes and the spectrum of half-BPS states in $N = 4$ supersymmetric string theory*, *Adv. Theor. Math. Phys.* **9** (2005) 527 [hep-th/0504005].
- [19] A. Sen, *Stretching the horizon of a higher dimensional small black hole*, *JHEP* **07** (2005) 073 [hep-th/0505122].

- [20] P. Kraus and F. Larsen, *Microscopic black hole entropy in theories with higher derivatives*, *JHEP* **09** (2005) 034 [[hep-th/0506176](#)].
- [21] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [[hep-th/0506177](#)].
- [22] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Precision counting of small black holes*, *JHEP* **10** (2005) 096 [[hep-th/0507014](#)].
- [23] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008 [[hep-th/0508042](#)].
- [24] S.D. Mathur, *The quantum structure of black holes*, *Class. and Quant. Grav.* **23** (2006) R115 [[hep-th/0510180](#)].
- [25] P. Prester, *Lovelock type gravity and small black holes in heterotic string theory*, *JHEP* **02** (2006) 039 [[hep-th/0511306](#)].
- [26] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Black hole partition functions and duality*, *JHEP* **03** (2006) 074 [[hep-th/0601108](#)].
- [27] A. Sinha and N.V. Suryanarayana, *Extremal single-charge small black holes: entropy function analysis*, *Class. and Quant. Grav.* **23** (2006) 3305 [[hep-th/0601183](#)].
- [28] F. Moura and R. Schiappa, *Higher-derivative corrected black holes: perturbative stability and absorption cross-section in heterotic string theory*, [hep-th/0605001](#).
- [29] A. Sinha and N.V. Suryanarayana, *Two-charge small black hole entropy: string-loops and multi-strings*, *JHEP* **10** (2006) 034 [[hep-th/0606218](#)].
- [30] B. Kol and A. Rajaraman, *Fixed scalars and suppression of Hawking evaporation*, *Phys. Rev. D* **56** (1997) 983 [[hep-th/9608126](#)].
- [31] S.R. Das, G.W. Gibbons and S.D. Mathur, *Universality of low energy absorption cross sections for black holes*, *Phys. Rev. Lett.* **78** (1997) 417 [[hep-th/9609052](#)].
- [32] E. Del Giudice, P. Di Vecchia and S. Fubini, *General properties of the dual resonance model*, *Ann. Phys. (NY)* **70** (1972) 378.
- [33] M. Karliner, I.R. Klebanov and L. Susskind, *Size and shape of strings*, *Int. J. Mod. Phys. A* **3** (1988) 1981.
- [34] R.R. Khuri, *Self-gravitating strings and string/black hole correspondence*, *Phys. Lett. B* **470** (1999) 73 [[hep-th/9910122](#)].
- [35] L.D. Landau and E.M. Lifshitz, *Quantum mechanics (course of theoretical physics, Vol. 3)* 3rd ed., New York, Pergamon Press (1975).